

# Topological rigidity

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# Topological rigidity

- The goal of this talk is to investigate when a closed topological manifold is topologically rigid in the following sense.

## Definition (Topological rigidity)

A closed topological manifold  $N$  is called **topologically rigid** if any homotopy equivalence  $f: M \rightarrow N$  with a closed manifold  $M$  as source and  $N$  as target is homotopic to a homeomorphism.

## Convention (Manifold)

*Manifold means in the following a connected closed orientable topological manifold.*

- On the first glance this seems to be a property which only very few manifolds may have. However, we will see that there are many more topological rigid manifolds than anticipated.
- Moreover, the problem to decide which topological manifolds are topologically rigid is very interesting, but also very difficult. It has lead to a lot of fruitful activities in topology and to interesting and sophisticated interactions of topology and other fields of mathematics.
- The goal of this talk is to illustrate the remarks above.

## Lemma

The *Poincaré Conjecture* in dimension  $n$  is equivalent to the assertion that  $S^n$  is topologically rigid.

## Proof.

Any self homotopy equivalence of a sphere is homotopic to a homeomorphism. □

- The existence of **exotic spheres** shows that in high dimension **smooth rigidity** is a very rare phenomenon.

## Dimension $\leq 3$

### Theorem (Dimension $\leq 2$ )

*Any manifold of dimension  $\leq 2$  is topologically rigid.*

### Theorem (Dimension 3)

*Every 3-manifold with torsionfree fundamental group is topologically rigid.*

- The main input in the proof are **Waldhausen's** rigidity results for Haken manifolds and the proof of **Thurston's** Geometrization Conjecture.
- Conclusion: If  $\pi_1(M)$  is torsionfree, then  $\pi_1(M)$  determines the homeomorphism type of a 3-manifold.
- The **lens space**  $L(7, 1, 1)$  is not topologically rigid.

# Very brief review of surgery theory

- The main tool to attack the question which manifolds of dimension  $\geq 4$  are topological rigid is **surgery theory**.
- A first introduction to surgery theory has already been given in previous talks on this conference.

We will concentrate on the **surgery exact sequence**. It is designed to compute the **structure set** of a manifold.

## Definition (The structure set)

Let  $N$  be a topological manifold of dimension  $n$ . We call two simple homotopy equivalences  $f_i: M_i \rightarrow N$  from closed topological manifolds  $M_i$  to  $N$  for  $i = 0, 1$  equivalent if there exists a homeomorphism  $g: M_0 \rightarrow M_1$  such that  $f_1 \circ g$  is homotopic to  $f_0$ .

$$\begin{array}{ccc} M_0 & \xrightarrow[\cong]{g} & M_1 \\ & \searrow f_0 & \swarrow f_1 \\ & N & \end{array}$$

The **structure set**  $\mathcal{S}_n^{\text{top}}(N)$  of  $N$  is the set of equivalence classes of simple homotopy equivalences  $M \rightarrow X$  from topological manifolds of dimension  $n$  to  $N$ .

This set has a preferred base point, namely the class of the identity  $\text{id}: N \rightarrow N$ .

- If we assume  $\text{Wh}(\pi_1(N)) = 0$ , then every homotopy equivalence with target  $N$  is automatically simple.
- There is an obvious version of the structure set, where topological and homeomorphism are replaced by smooth and diffeomorphism.



## Lemma

*A topological manifold  $M$  is topologically rigid if and only if the structure set  $\mathcal{S}_n^{\text{top}}(M)$  consists of exactly one point.*

## Proof.

- Suppose that the structure set consists of one element. Consider any simple homotopy equivalence  $f: M \rightarrow N$  with  $N$  as target. Since  $[f] = [\text{id}_N]$ , there exists a homeomorphism  $g: M \rightarrow N$  with  $\text{id}_N \circ g \simeq f$  and hence with  $g \simeq f$ . This shows that  $N$  is topologically rigid.
- Suppose that  $N$  is topologically rigid. Let  $[f: M \rightarrow N]$  be an element in the structure set. Choose a homeomorphism  $g: M \rightarrow N$  with  $g \simeq f$ . Hence  $\text{id}_N \circ g \simeq f$  which implies  $[f] = [\text{id}_N]$ .



## Definition (Normal map of degree one)

A **normal map of degree one** with target a finite CW-complex  $X$  consists of:

- An (oriented)  $n$ -dimensional manifold  $M$ ;
- A map of degree one  $f: M \rightarrow X$ ;
- A  $(k + n)$ -dimensional vector bundle  $\xi$  over  $X$ ;
- A bundle map  $\bar{f}: TM \oplus \underline{\mathbb{R}^k} \rightarrow \xi$  covering  $f$ .

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^k} & \xrightarrow{\bar{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

## Problem (Surgery Problem)

Let  $(f, \bar{f}): M \rightarrow X$  be a normal map of degree one. Can we modify it without changing the target such that  $f$  becomes a (simple) homotopy equivalence?

- By surgery one can always arrange that  $f$  is  $k$ -connected, where  $n = 2k$  or  $n = 2k + 1$ .
- Suppose that  $X$  is homotopy equivalent to a manifold  $M$ .
- Then there exists a normal map of degree one from  $M$  to  $X$  whose underlying map  $f: M \rightarrow X$  is a homotopy equivalence. Just take  $\xi = f^{-1}TM$  for some homotopy inverse  $f^{-1}$  of  $f$ .
- The finite CW-complex  $X$  has to be a Poincaré complex in the following sense, if the surgery problem can be solved.

## Definition (Finite Poincaré complex)

A connected finite  $n$ -dimensional CW-complex  $X$  is a (simple) finite  $n$ -dimensional Poincaré complex if there is  $[X] \in H_n(X; \mathbb{Z}^w)$  such that the induced  $\mathbb{Z}\pi$ -chain map

$$-\cap [X]: C^{n-*}(\tilde{X}) \rightarrow C_*(\tilde{X})$$

is a (simple)  $\mathbb{Z}\pi$ -chain homotopy equivalence.

## Theorem (Manifolds are Poincaré complexes)

*A closed  $n$ -dimensional manifold  $M$  is a simple finite  $n$ -dimensional Poincaré complex with  $w = w_1(X)$ .*

- One can assign to a finite Poincaré complex its so called **Spivak normal spherical fibration**.
- If  $M$  is a manifold  $M$  with normal bundle  $\nu$ , then the Spivak normal spherical fibration of  $M$  regarded as a finite Poincaré complex is given by the sphere bundle of  $\nu$ .
- Hence a necessary condition for a finite Poincaré complex to be homotopy equivalent to a manifold is that its Spivak normal spherical fibration has a **vector bundle reduction**, i.e., is the sphere bundle of an appropriate vector bundle.
- There are finite Poincaré complexes which do not admit such a reduction.
- If a finite Poincaré complex admits such a reduction, then there exists a normal map with  $X$  as target.

- From now on  $X$  is a simple finite Poincaré complex of dimension  $n$ .

## Lemma

*A normal map of degree one which is  $(k + 1)$ -connected, where  $n = 2k$  or  $n = 2k + 1$ , is a homotopy equivalence.*

## Proof.

- It suffices to show by the Hurewicz Theorem and the Whitehead Theorem, that  $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$  induces on all homology groups an isomorphism.
- This follows from the assumption about its connectivity and Poincaré duality.



- Hence we have to make a normal map, which is already  $k$ -connected,  $(k + 1)$ -connected in order to achieve a homotopy equivalence, where  $n = 2k$  or  $n = 2k + 1$ . Exactly here the **surgery obstruction** occurs.
- In odd dimension  $n = 2k + 1$  the surgery obstruction comes from the observation that by Poincare duality modifications in the  $(k + 1)$ -th homology cause automatically (undesired) changes in the  $k$ -th homology.
- In even dimension  $n = 2k$  one encounters the problem that the bundle data only guarantee that one can find an immersion with finitely many self-intersection points

$$q^{\text{th}}: S^k \times D^k \rightarrow M.$$

The surgery obstruction is the algebraic obstruction to get rid of the self-intersection points. If  $n \geq 5$ , its vanishing is indeed sufficient to convert  $q^{\text{th}}$  into an embedding.

- One prominent necessary surgery obstruction is given in the case  $n = 4k$  by the difference of the **signatures**  $\text{sign}(X) - \text{sign}(M)$  since the signature is a bordism invariant and a homotopy invariant.
- If  $\pi_1(M)$  is simply connected and  $n = 4k$  for  $k \geq 2$ , then the vanishing of  $\text{sign}(X) - \text{sign}(M)$  is indeed sufficient.
- If  $\pi_1(M)$  is simply connected and  $n$  is odd and  $n \geq 5$ , there are no surgery obstructions.



## Theorem (Surgery obstruction)

- For a normal map  $(f, \bar{f})$  with a simple finite Poincaré complex  $X$  of dimension  $n$  as target, there exists a surgery obstruction

$$\sigma(f, \bar{f}) \in L_n^S(\mathbb{Z}[\pi_1(X)]);$$

- It is a normal bordism invariant;
- Its vanishing is a necessary condition for changing  $f$  by surgery into a simple homotopy equivalence;
- Its vanishing is a sufficient condition for changing  $f$  by surgery into a simple homotopy equivalence, provided that  $n \geq 5$ .

## Theorem (The topological Surgery Exact Sequence)

For a  $n$ -dimensional topological manifold  $N$  with  $n \geq 5$ , there is an exact sequence of abelian groups, called *surgery exact sequence*,

$$\begin{aligned} \dots \xrightarrow{\eta} \mathcal{N}_{n+1}^{\text{top}}(N \times [0, 1], N \times \{0, 1\}) \xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi) \xrightarrow{\partial} \mathcal{S}_n^{\text{top}}(N) \\ \xrightarrow{\eta} \mathcal{N}_n^{\text{top}}(N) \xrightarrow{\sigma} L_n^s(\mathbb{Z}\pi). \end{aligned}$$

- $L_n^s(\mathbb{Z}\pi)$  is the algebraic  $L$ -group of the group ring  $\mathbb{Z}\pi$  for  $\pi = \pi_1(N)$  (with decoration  $s$ ).
- $\mathcal{N}_n^{\text{top}}(N)$  is the set of normal bordism classes of normal maps of degree one with target  $N$ .
- $\mathcal{N}_{n+1}^{\text{top}}(N \times [0, 1], N \times \{0, 1\})$  is the set of normal bordism classes of normal maps  $(M, \partial M) \rightarrow (N \times [0, 1], N \times \{0, 1\})$  of degree one with target  $N \times [0, 1]$  which are simple homotopy equivalences on the boundary.

- The map  $\sigma$  is given by the surgery obstruction.
- The map  $\eta$  sends  $f: M \rightarrow N$  to the normal map of degree one for which  $\xi = (f^{-1})^* TN$ .
- The map  $\partial$  sends an element  $x \in L_{n+1}(\mathbb{Z}\pi)$  to  $f: M \rightarrow N$  if there exists a normal map  $F: (W, \partial W) \rightarrow (N \times [0, 1], N \times \{0, 1\})$  of degree one with target  $N \times [0, 1]$  such that  $\partial W = N \amalg M$ ,  $F|_N = \text{id}_N$ ,  $F|_M = f$ , and the surgery obstruction of  $F$  is  $x$ .

- There is a space  $\mathbf{G}/\mathbf{TOP}$  together with bijections

$$\begin{aligned}
 [N, \mathbf{G}/\mathbf{TOP}] &\xrightarrow{\cong} \mathcal{N}_n^{\text{top}}(N); \\
 [N \times [0, 1]/N \times \{0, 1\}, \mathbf{G}/\mathbf{TOP}] &\xrightarrow{\cong} \mathcal{N}_{n+1}^{\text{top}}(N \times [0, 1], N \times \{0, 1\}).
 \end{aligned}$$

- We have

$$\begin{aligned}
 [X, \mathbf{G}/\mathbf{TOP}] \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] &\cong \widetilde{KO}^0(X) \left[ \begin{array}{c} 1 \\ 2 \end{array} \right]; \\
 [X, \mathbf{G}/\mathbf{TOP}]_{(2)} &\cong \prod_{j \geq 1} H^{4j}(X; \mathbb{Z}/(2)) \times \prod_{j \geq 1} H^{4j-2}(M; \mathbb{Z}/(2)),
 \end{aligned}$$

where  $KO^*$  is K-theory of real vector bundles.

# The Borel Conjecture

## Conjecture (Borel Conjecture)

The *Borel Conjecture for  $G$*  predicts that an aspherical manifold  $M$  satisfying  $\pi_1(M) \cong G$  is topological rigid.

- If  $G$  satisfies the Borel Conjecture, then two aspherical manifolds whose fundamental groups are isomorphic to  $G$ , are homeomorphic.
- This is the topological version of *Mostow rigidity*. One version of Mostow rigidity says that any homotopy equivalence between hyperbolic Riemannian manifolds of dimension  $\geq 3$  is homotopic to an isometric diffeomorphism. In particular they are isometrically diffeomorphic if and only if their fundamental groups are isomorphic.

- The Borel Conjecture is in general false in the smooth category. For instance, for  $n \geq 5$ , there exists a smooth manifold  $M$  which is homeomorphic but not diffeomorphic to  $T^n$ .
- In some sense the Borel Conjecture is opposed to the **Poincaré Conjecture**. Namely, in the Borel Conjecture the fundamental group can be complicated but there are no higher homotopy groups, whereas in the Poincaré Conjecture there is no fundamental group but complicated higher homotopy groups.

# The Farrell-Jones Conjecture

Conjecture (*L-theoretic Farrell-Jones Conjecture for torsionfree groups*)

The *L-theoretic Farrell-Jones Conjecture* predicts that the *assembly map*

$$H_n(BG; \mathbf{L}_{\mathbb{Z}}) \rightarrow L_n^s(\mathbb{Z}G)$$

is bijective for all  $n \in \mathbb{Z}$ .

- $\mathbf{L}_{\mathbb{Z}}$  is the so called *L-theory spectrum*. It satisfies

$$\pi_n(\mathbf{L}_{\mathbb{Z}}) \cong L_n^s(\mathbb{Z}) \cong L_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n \equiv 0 \pmod{4}; \\ \mathbb{Z}/2 & n \equiv 2 \pmod{4}; \\ \{0\} & n \text{ odd.} \end{cases}$$

- There is also a  $K$ -theoretic version of the Farrell-Jones Conjecture which implies for a torsionfree group  $G$  that  $K_n(\mathbb{Z}G)$  for  $n \leq -1$ ,  $\tilde{K}_0(\mathbb{Z}G)$ , and  $\text{Wh}(G)$  vanish.
- Therefore it does not matter which decoration we use for the  $L$ -groups and any homotopy equivalence is a simple homotopy equivalence, if  $G$  satisfies the Farrell-Jones Conjecture.



## Theorem (Bartels, Bestvina, Farrell, Kammeyer, Lück, Reich, Rüping, Wegner)

Let  $\mathcal{FJ}$  be the class of groups for which the (Full) Farrell-Jones Conjecture holds. Then  $\mathcal{FJ}$  contains the following groups:

- Hyperbolic groups;
- CAT(0)-groups;
- Solvable groups,
- (Not necessarily uniform) lattices in almost connected Lie groups;
- Fundamental groups of (not necessarily compact)  $d$ -dimensional manifolds (possibly with boundary) for  $d \leq 3$ .
- Subgroups of  $GL_n(\mathbb{Q})$  and of  $GL_n(F[t])$  for a finite field  $F$ .
- All  $S$ -arithmetic groups.
- mapping class groups.

## Theorem (continued)

Moreover,  $\mathcal{FJ}$  has the following inheritance properties:

- If  $G_1$  and  $G_2$  belong to  $\mathcal{FJ}$ , then  $G_1 \times G_2$  and  $G_1 * G_2$  belong to  $\mathcal{FJ}$ ;
- If  $H$  is a subgroup of  $G$  and  $G \in \mathcal{FJ}$ , then  $H \in \mathcal{FJ}$ ;
- If  $H \subseteq G$  is a subgroup of  $G$  with  $[G : H] < \infty$  and  $H \in \mathcal{FJ}$ , then  $G \in \mathcal{FJ}$ ;
- Let  $\{G_i \mid i \in I\}$  be a directed system of groups (with not necessarily injective structure maps) such that  $G_i \in \mathcal{FJ}$  for  $i \in I$ . Then  $\operatorname{colim}_{i \in I} G_i$  belongs to  $\mathcal{FJ}$ ;
- Let  $1 \rightarrow K \rightarrow G \xrightarrow{p} Q \rightarrow 1$  be an exact sequence. Suppose that  $Q$  and  $p^{-1}(V)$  for every virtually cyclic subgroup  $V \subseteq Q$  belong to  $\mathcal{FJ}$ . Then also  $G$  belongs to  $\mathcal{FJ}$ .

## Theorem (The Farrell-Jones Conjecture implies the Borel Conjecture)

*If the  $K$ -theoretic and the  $L$ -theoretic Farrell-Jones Conjecture hold for the group  $G$ , then the Borel Conjecture holds for any  $n$ -dimensional aspherical manifold with  $\pi_1(M) \cong G$ , provided that  $n \geq 5$ .*

- Next we sketch the proof.

## Theorem (Algebraic surgery sequence, Ranicki)

There is an exact sequence of abelian groups called *algebraic surgery exact sequence* for an  $n$ -dimensional manifold  $M$

$$\begin{array}{ccccccc} \dots & \xrightarrow{\sigma_{n+1}} & H_{n+1}(M; \mathbf{L}_{\mathbb{Z}}\langle 1 \rangle) & \xrightarrow{A_{n+1}} & L_{n+1}(\mathbb{Z}\pi_1(M)) & \xrightarrow{\partial_{n+1}} & \\ & & & & \mathcal{S}^{\text{top}}(M) & \xrightarrow{\sigma_n} & H_n(M; \mathbf{L}_{\mathbb{Z}}\langle 1 \rangle) & \xrightarrow{A_n} & L_n(\mathbb{Z}\pi_1(M)) & \xrightarrow{\partial_n} & \dots \end{array}$$

It can be identified with the classical geometric surgery exact sequence due to *Sullivan and Wall* in high dimensions.

- Here  $\mathbf{L}_{\mathbb{Z}}\langle 1 \rangle$  is the **1-connective cover** of the  $L$ -theory spectrum  $\mathbf{L}_{\mathbb{Z}}$ . It comes with a natural map of spectra  $\mathbf{i}: \mathbf{L}_{\mathbb{Z}}\langle 1 \rangle \rightarrow \mathbf{L}_{\mathbb{Z}}$  which induces on  $\pi_i^{\mathcal{S}}$  an isomorphism for  $i \geq 1$ , and we have  $\pi_i^{\mathcal{S}}(\mathbf{L}_{\mathbb{Z}}\langle 1 \rangle) = 0$  for  $i \leq 0$ .

- There are natural identifications

$$\mathcal{N}(M) \cong [M, \mathbf{G}/\mathbf{TOP}] \cong H_n(M; \mathbf{L}_{\mathbb{Z}}\langle 1 \rangle).$$

- We can write  $A_n$  as the composite

$$A_n: \mathcal{N}(M) = H_n(M; \mathbf{L}_{\mathbb{Z}}\langle 1 \rangle) \xrightarrow{H_n(\text{id}_M; \mathbf{i})} H_n(M; \mathbf{L}_{\mathbb{Z}}) = H_n(B\mathbf{G}; \mathbf{L}_{\mathbb{Z}}) \rightarrow L_n(\mathbb{Z}\mathbf{G}).$$

- The map  $A_n$  can be identified with the map given by the surgery obstruction in the geometric surgery exact sequence.
- This gives an interesting **interpretation** of the homotopy theoretic assembly map in geometric terms. Its proof is non-trivial.
- The analog statement about  $A_m$  holds in all degrees  $m \geq n$ .

- $\mathcal{S}^{\text{top}}(M)$  consists of one element if and only if  $A_{n+1}$  is surjective and  $A_n$  is injective.
- An easy spectral sequence argument shows that

$$H_m(\text{id}_M; \mathbf{i}): H_m(M; \mathbf{L}_{\mathbb{Z}}\langle 1 \rangle) \rightarrow H_m(M; \mathbf{L}_{\mathbb{Z}})$$

is bijective for  $m \geq n + 1$  and injective for  $m = n$ .

- This finishes the proof, since the Farrell-Jones Conjecture implies for  $m = n, n + 1$  the bijectivity of

$$H_n(M; \mathbf{L}_{\mathbb{Z}}) = H_n(BG; \mathbf{L}_{\mathbb{Z}}) \rightarrow L_n(\mathbb{Z}G).$$

### Theorem (Chang-Weinberger)

Let  $M^{4k+3}$  be a manifold for  $k \geq 1$  whose fundamental group has torsion.

Then there are infinitely many pairwise not homeomorphic smooth manifolds which are homotopy equivalent to  $M$  but not homeomorphic to  $M$ .

In particular  $M$  is not topologically rigid.

- The proof of the result uses kind of  **$L^2$ -Rho-invariants** in order to show that the action of  $L_{4k+4}^S(\mathbb{Z}\pi)$  on  $\mathcal{S}_n(M)$  produces infinitely many elements  $[f: M_i \rightarrow M]$  such that  $M_i$  and  $M_j$  are not diffeomorphic if  $i \neq j$ .
- This result confirms the observation that the fundamental groups of all known topologically rigid manifolds are torsionfree.

## Theorem (Connected sums, Kreck-Lück)

*Let  $M$  and  $N$  be manifolds of the same dimension  $n \geq 5$  such that neither  $\pi_1(M)$  nor  $\pi_1(N)$  contains elements of order 2 or that  $n = 0, 3 \pmod{4}$ .*

*If both  $M$  and  $N$  are topologically rigid, then the same is true for their connected sum  $M \# N$ .*

- The proof is based on Cappell's work on splitting obstructions and of UNIL-groups and improvements by Banagl, Connolly, Davis, Ranicki.
- Notice in general connected sums of two manifolds are not aspherical.



## Theorem (Products of two spheres)

Suppose that  $k + d \neq 3$ . Then  $S^k \times S^d$  is topologically rigid if and only if both  $k$  and  $d$  are odd.

### Proof.

- We only treat the case  $k, d \geq 2$  and  $k + d \geq 5$ . Then  $S^k \times S^d$  is simply-connected.
- The structure set can be computed by

$$a_1 \times a_2: \mathcal{S}^{\text{top}}(S^k \times S^d) \xrightarrow{\cong} L_k(\mathbb{Z}) \oplus L_d(\mathbb{Z}),$$

where  $a_1$  and  $a_2$  respectively send the class of a homotopy equivalence  $f: M \rightarrow S^k \times S^d$  to the surgery obstruction of the surgery problem with target  $S^k$  and  $S^d$  respectively which is obtained from  $f$  by making it transversal to  $S^k \times \text{pt}$  and  $\text{pt} \times S^d$  respectively.

## Theorem (Homology spheres, Kreck-Lück)

Let  $M$  be a manifold of dimension  $n \geq 5$  with fundamental group  $\pi = \pi_1(M)$ .

- Let  $M$  be an integral homology sphere. Then  $M$  is topologically rigid if and only if

$$L_{n+1}^S(\mathbb{Z}) \xrightarrow{\cong} L_{n+1}^S(\mathbb{Z}\pi);$$

- Suppose that  $M$  is a rational homology sphere and topologically rigid. Suppose that  $\pi$  satisfies the Farrell-Jones Conjecture. Then

$$H_{n+1-4i}(B\pi; \mathbb{Q}) = 0$$

for  $i \geq 1$  and  $n + 1 - 4i \neq 0$ .

## Theorem (Another construction of topologically rigid manifolds, Kreck-Lück)

*Start with a topologically rigid manifold  $M$  of dimension  $n \geq 5$ . Choose an embedding  $S^1 \times D^{n-1} \rightarrow M$  which induces an injection on  $\pi_1$ . Choose a high dimensional knot  $K \subseteq S^n$  with complement  $X$  such that the inclusion  $\partial X \cong S^1 \times S^{n-2} \rightarrow X$  induces an isomorphism on  $\pi_1$ . Put*

$$M' := M - (S^1 \times D^{n-1}) \cup_{S^1 \times S^{n-2}} X.$$

*Then  $M'$  is topologically rigid.*

- If  $M$  is aspherical, then  $M'$  is in general not aspherical.